Previous chapter:

interference from waves thru 2 slits

⇒ wavelength $\lambda \sim$ same or bigger than slit <u>width</u> so can treat waves in slits as single Huygens wavelets

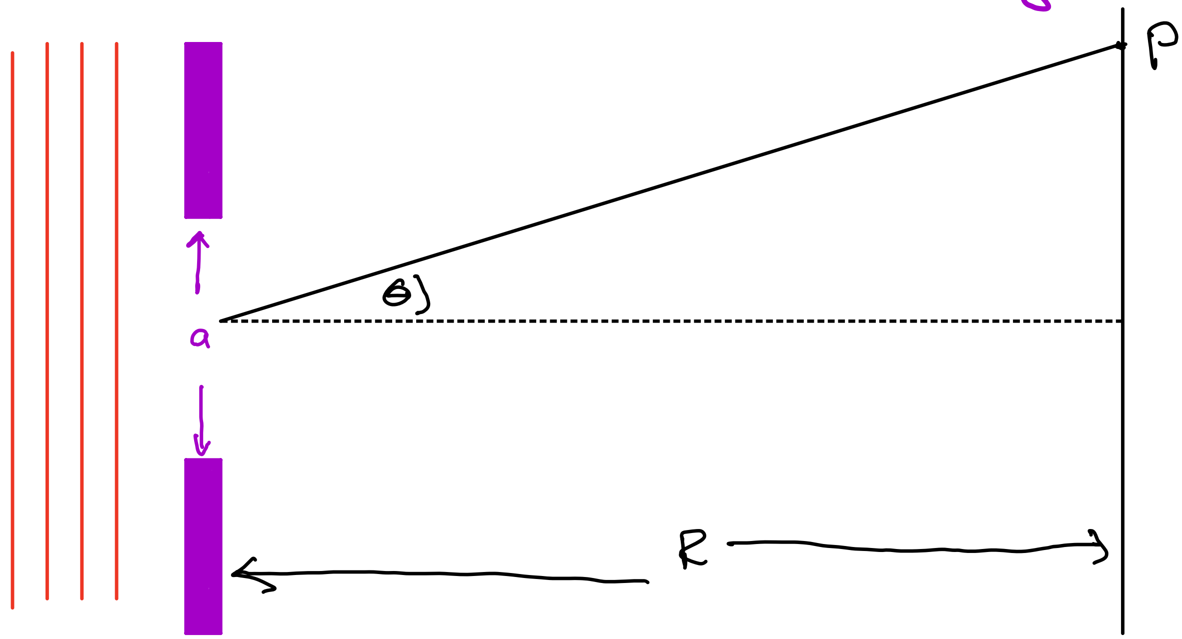
⇒ what if $\lambda \ll$ slit width?
this is the case with light
(hard to make slit $\lesssim 1\,\mu m$!)

Then apply Huygen's principle to all the wavelets that fit inside slit

This is called <u>diffraction</u>

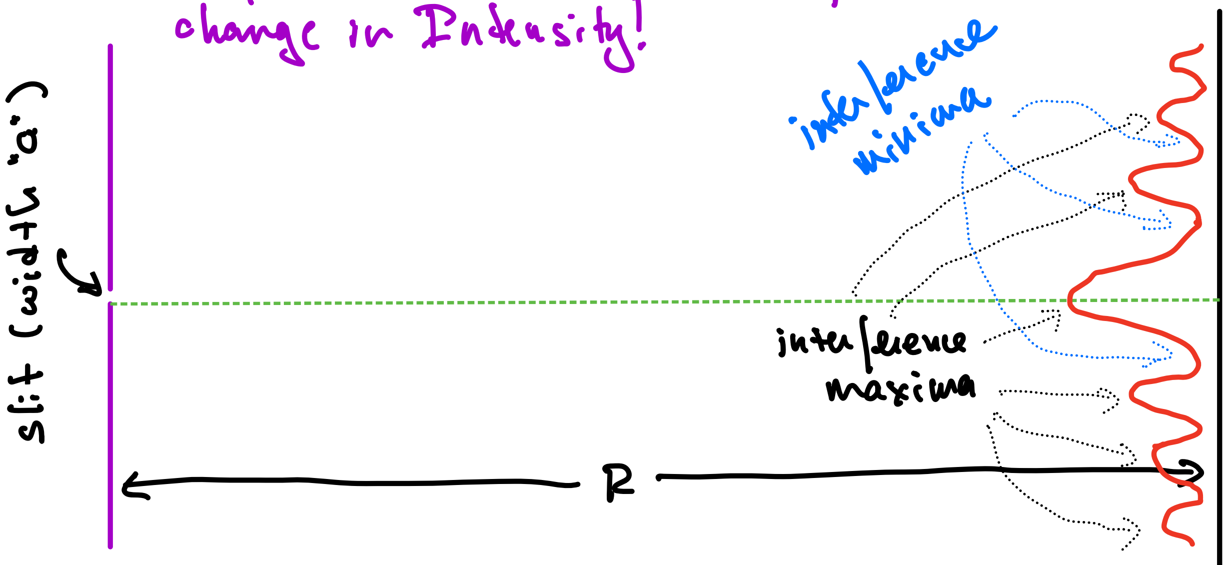
Franhofer diffraction:

when slit and screen are separated by $R > a$



want to calculate the diffraction (interference) minima and maxima at point P

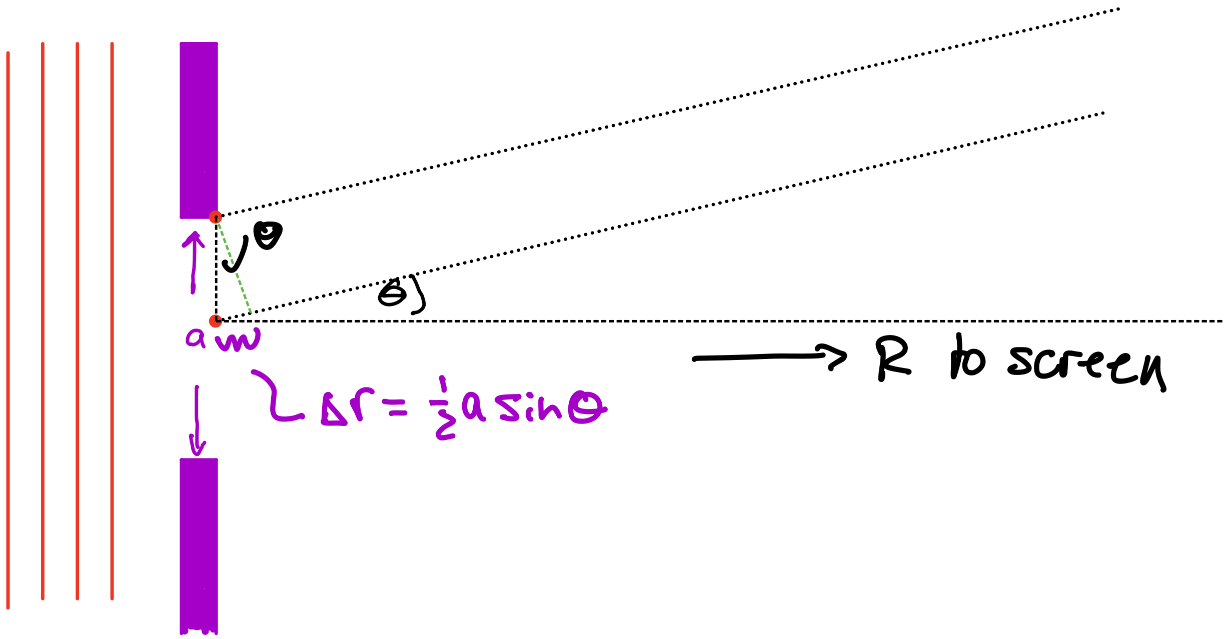
From experiment we see this pattern - note the change in Intensity!



Find 1st minima above central maxima:

1. Divide the slit into 2 parts
2. Draw a ray from top & middle to point on screen (a distance $R \gg a$ away) where 1st minima is

   $\Rightarrow$ each pair of rays will be parallel (same θ)
3. path difference is $\frac{1}{2}a \cdot \sin\theta$ just like for interference calculation



4. These 2 rays will cause destructive interference at the point on screen that is at θ (with respect to horizontal) if path difference is $\Delta r = \frac{1}{2}\lambda$

   so $\frac{1}{2} a \sin\theta = \frac{1}{2}\lambda$

   or $a \sin\theta = \lambda$

so 1st minimum is at $a \sin\theta = \lambda$
for this pair

- But also for the pair of rays just below this first pair
- and for all pairs

so 1st minimum is at θ_1 where $a \sin\theta_1 = \lambda$

Next minimum is 1 λ more of a path diff.

so θ_2 given by $a \sin\theta_2 = 2\lambda$

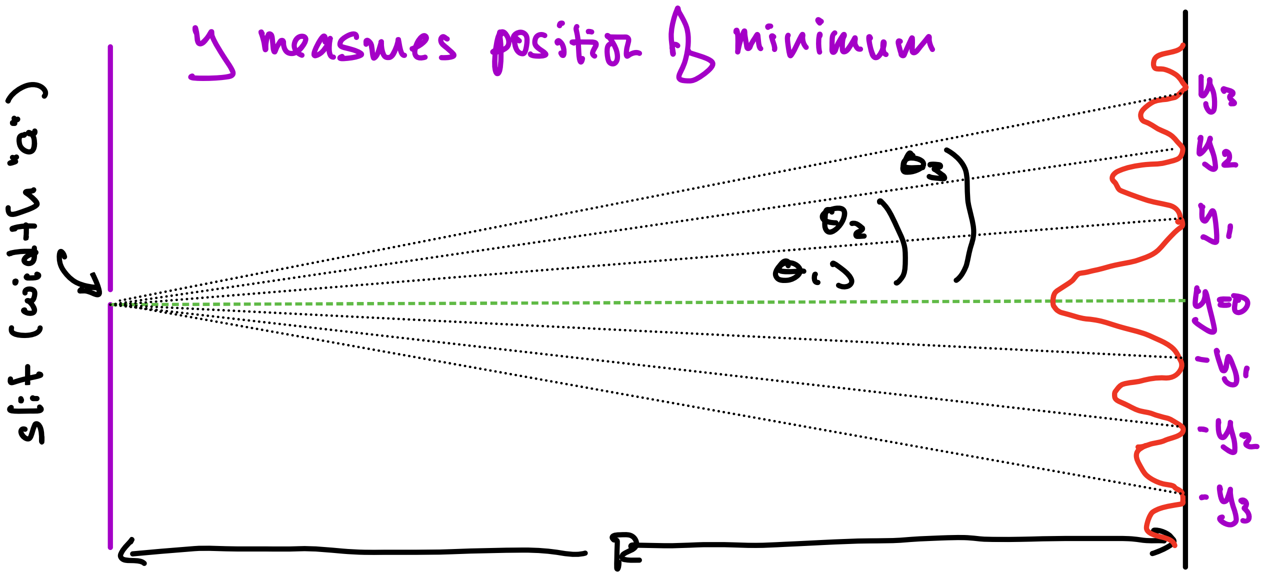
so in general: n^{th} minimum is given by

$$\boxed{a \sin\theta_n = n\lambda} \qquad n = \pm1, \pm2, \pm3, \ldots$$

why $\pm$? because $n=+1$ is 1st min above central maximum

so $n=-1$ is 1st min below central max

y measures position of minimum



$\tan\theta_n = \frac{y_n}{R}$

for small angles, $\tan\theta = \frac{\sin\theta}{\cos\theta} \sim \sin\theta$ because $\cos\theta \to 1$ as $\theta \to 0$

so $\sin\theta_n = \frac{y_n}{R}$

and $a\sin\theta_n = n\lambda$

so $a\frac{y_n}{R} = n\lambda \Rightarrow \boxed{y_n = n\frac{\lambda R}{a}}$

At $n=0$, no path difference so that points to central maximum

ex: let $\lambda = 400\,nm$
$a = 1\,mm$
$R = 1\,m$

1. how many diffraction minima are there up to $\pm 1\,cm$ on screen?

- at $y = 1cm$, solve for n

$$10^{-2}m = n \cdot \frac{400\times10^{-9}m \cdot 1m}{10^{-3}m}$$

$$n = \frac{10^{-2}\cdot10^{-3}}{400\times10^{-9}} = \frac{10^{4}}{400} = 25$$

so there are 25 min above and 25 below
total of 50

2. what is width of central max?

1st min above center of central max:

200 3/10

$$y_1 = \frac{n\lambda R}{a} = \frac{1 \cdot 400\times10^{-9} \cdot 1}{10^{-3}}$$

$$= 400\times10^{-6}\,m = 400\,\mu m$$

1st min below is at $y = -400\mu m$

so width of central max $\omega = 800\mu m$
$= 0.8\,mm$

ex: light of wavelength 570nm on slit
screen is $R = 7.5\,m$ away
width of central max is 3.2cm

how wide is slit?

1st minima is $\frac{3.2cm}{2} = 1.6cm$ above center of central max

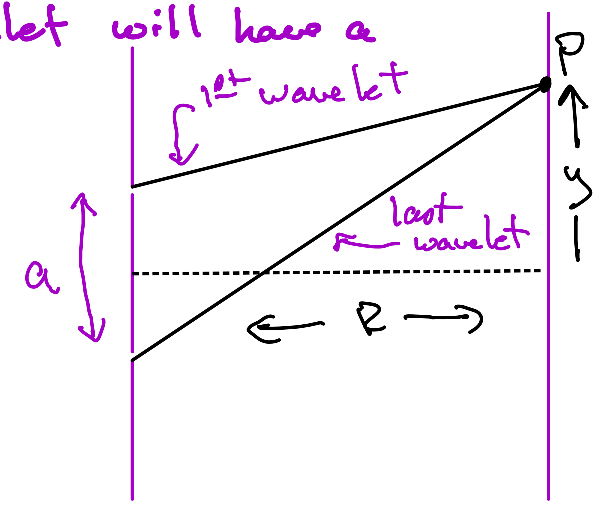
$$\sin\theta \approx \tan\theta = \frac{1.6cm}{7.5m} = 0.00213 = \frac{n\lambda}{a} \quad n=1$$

$$\text{so } a = \frac{\lambda}{.00213} = \frac{.570\times10^{-6}\,m}{2.13\times10^{-3}} = 0.27\,mm$$

Intensity pattern

Each wavelet has an E field ⇒ E_n n=wavelet

At point P, each wavelet will have a phase shift from having different path length

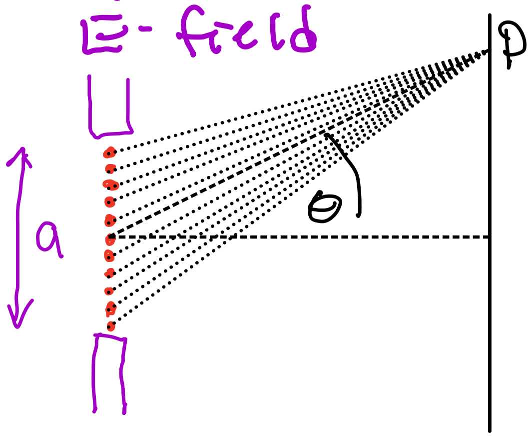


Add up all the waves as vectors

But assume $R \gg a$

⇒ all waves are ~ parallel

so just need to add amplitudes to get final E-field



E_{TOT} at P = sum E at each wavelet

⇒ each E will have slightly different phase at point P due to the different path length

(Textbook derivation is pretty good - also see below)

result: $I = I_0 \frac{\sin^2(x)}{x^2}$ where $x = \frac{\pi a \sin\theta}{\lambda}$

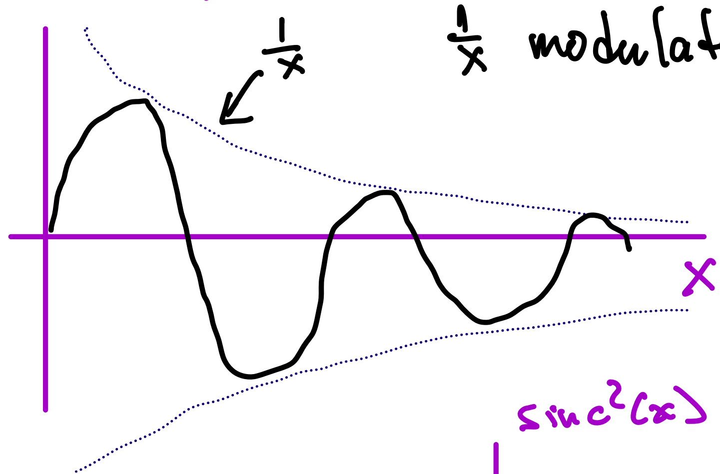
where θ = angle between dashed horizontal line (from cent of slit to screen) and wavelet ray from center of slit to P

I_0 is intensity at central maximum: $\theta = 0$

note: $\frac{\sin(x)}{x} \equiv$ "sinc function"

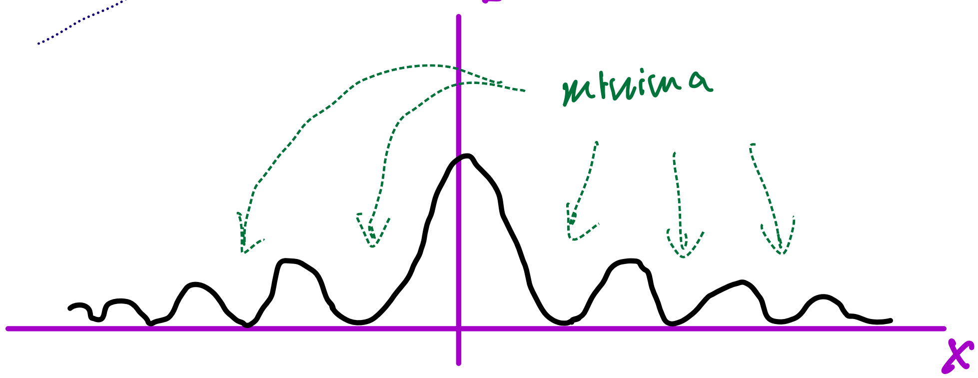
also $\frac{\sin(x)}{x}$ as $x \to 0 = 1$

$\text{sinc}(x) = \frac{1}{x}\sin(x)$



$\frac{1}{x}$ modulates amplitude of sine term

$\text{sinc}^2(x)$



Intensity minima: at $\sin(x) = 0$

so $\quad x = \pi \dfrac{a \sin\theta}{\lambda} = m\pi$

$a \sin\theta = m\lambda$ as before

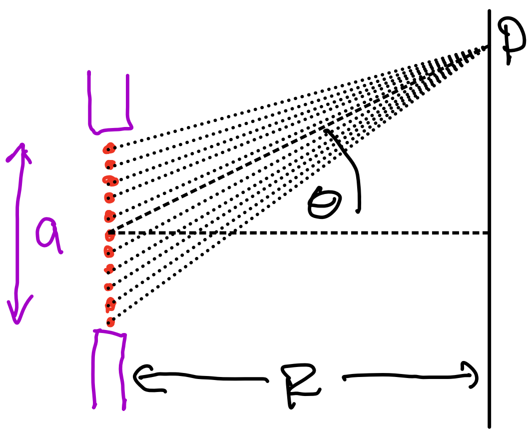
Intensity pattern derivation

To do this right we would pick a point P and add up the interference from all wavelets

First calculate net electric field

$\Rightarrow$ Each wavelet has an E-field that has this form: $E_i = E_0 \cos(kr - \omega t + \phi_i)$

where "i" labels the phase of the wavelet at point P.



$E_T = \sum_i E_i$ sum over wavelets

As # wavelets $\rightarrow \infty$, sum turns into an integral:

$$E_T = \frac{1}{a}\int_{-a/2}^{a/2} E_i \, dx$$

the integral goes from $-\frac{a}{2}$ to $\frac{a}{2}$ because we define the phase difference relative to the central wavelet

let x be the distance above the center of the slit, and $-\frac{a}{2} \leq x \leq +\frac{a}{2}$

the phase difference between central wavelet and any other wavelet at coordinate "x"

is as usual: $\phi = k\Delta r$

where Δr = difference in distance from the 2 wavelets to point P

just as w/ 2-slit interference:

$$\Delta r = x \sin\theta$$

so $\phi_i = k\Delta r_i = k x_i \sin\theta$

$$= 2\pi \frac{x_i \sin\theta}{\lambda}$$

integral is:

$$E_{TOT} = \frac{1}{a}\int_{-a/2}^{a/2} E_0 \cos\left(kr - \omega t + \frac{2\pi \sin\theta}{\lambda} x\right) dx$$

- the $\frac{1}{a}$ is needed to cancel out the added position dimension from integrating over x
- or you could think of the integral as being over the fractional distance $d\left(\frac{x}{a}\right)$

this integral is easy:

$$E_{TOT} = \frac{1}{a} E_0 \left. \frac{\sin\left(kr - \omega t + \frac{2\pi}{\lambda}\sin\theta\, x\right)}{\frac{2\pi \sin\theta}{\lambda}} \right|_{-a/2}^{a/2}$$

$$= \frac{E_0 \lambda}{2\pi a \sin\theta} \left[\sin\left(kr - \omega t + \frac{\pi a}{\lambda}\sin\theta\right) - \sin\left(kr - \omega t - \frac{\pi a}{\lambda}\sin\theta\right)\right]$$

let $A = kr - \omega t$, $B = \frac{\pi a}{\lambda}\sin\theta$

then $\sin(A+B) - \sin(A-B)$

$$= \sin A \cos B + \cos A \sin B - (\sin A \cos B - \cos A \sin B)$$

$$= 2 \cos A \sin B$$

so $E_{TOT} = E_0 \cos(kr - \omega t) \dfrac{\sin\left(\frac{\pi a}{\lambda}\sin\theta\right)}{\frac{\pi a \sin\theta}{\lambda}}$

$$\frac{\sin\left(\frac{\pi a}{\lambda}\sin\theta\right)}{\frac{\pi a \sin\theta}{\lambda}} \equiv \text{sinc}(x) \Rightarrow \text{sinc}(x) = \frac{\sin(x)}{x}$$

so $E_{TOT} = E_0 \cos(kr - \omega t)\, \text{sinc}\left(\frac{\pi a \sin\theta}{\lambda}\right)$

then intensity

$$I = \varepsilon_0 E_{TOT}^2 c$$

$$= \underbrace{\epsilon_0 c E_0^2 \cos^2(kr-\omega t)}_{I_0} \operatorname{sinc}^2\left(\frac{\pi a \sin\theta}{\lambda}\right)$$

$$I = I_0 \operatorname{sinc}^2\left(\frac{\pi a \sin\theta}{\lambda}\right)$$

1^{st} min is where $\frac{\pi a \sin\theta}{\lambda} = \pi$ $(\sin(\pi) = 0)$

or $a\sin\theta = \lambda$

2^{nd} min is when $\frac{\pi a \sin\theta}{\lambda} = 2\pi$

or $a\sin\theta = 2\lambda$

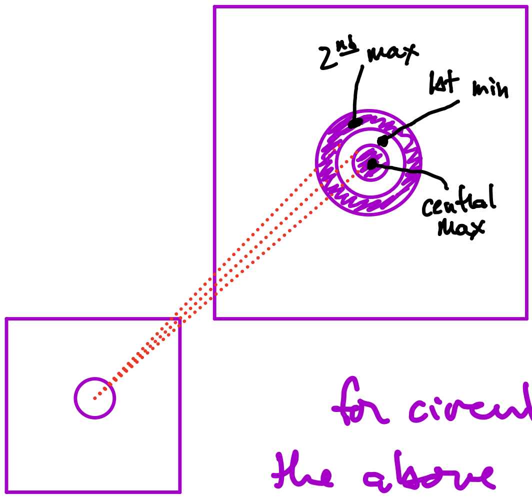
etc: $a\sin\theta = \pm m\lambda$ $m = 1, 2, 3 \ldots$ minima

We don't look for maxima this way because the sinc function is the product of 2 functions: $\operatorname{sinc}(x) = \frac{1}{x} \cdot \sin(x)$

max of $\sin(x)$ is not necessarily max of sinc

$\Rightarrow$ but min sinc = min sin !

Circular slits also forms diffraction patterns



central max is called "Airy's disc"

for circular apertures (slits) the above integral is more complicated

condition for 1st minima:

$$D \sin\theta = 1.22\lambda$$

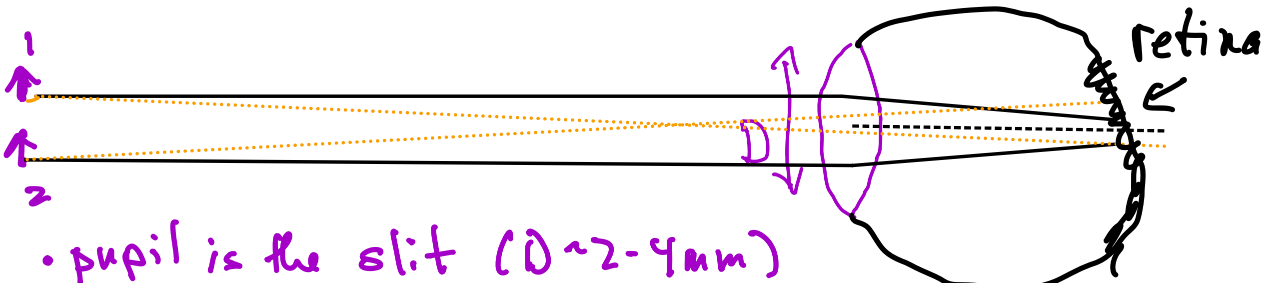
(1.22 comes from the complicated integral)

diameter of aperture

angle between central & 2nd maxima

⇒ so objects will not form crisp images on the retina or film due to diffraction spreading image out

⇒ diffraction will then cause images from 2 objects that are close together to overlap → want central maxima to be far apart as possible so that central maxima don't overlap (blurred together)

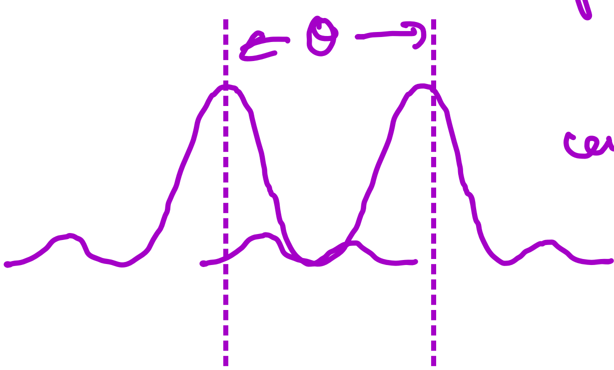


- pupil is the slit (D ~ 2-4 mm)
- each object produces diffraction around image
- if objects are too close together, diffraction will blur them on retina so image will be blurry

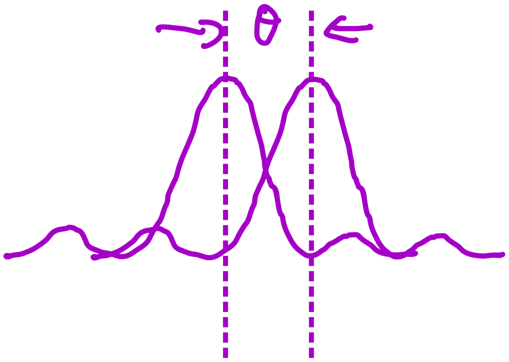
⇒ let θ be the angular separation between the images → this is also the angular separation of the objects

if $\sin\theta < 1.22\frac{\lambda}{D}$ then the image of one will fall in the Airy disc of the other

⇒ images are not resolvable!

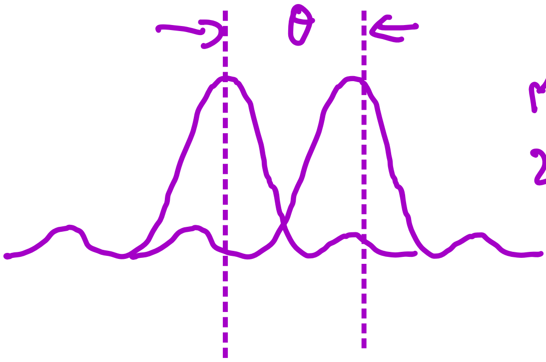


central max are far enough apart so you see the images clearly



central max's are close - max of 1 falls on min of the other - barely resolvable!

here $\theta = \frac{1.22\lambda}{D}$



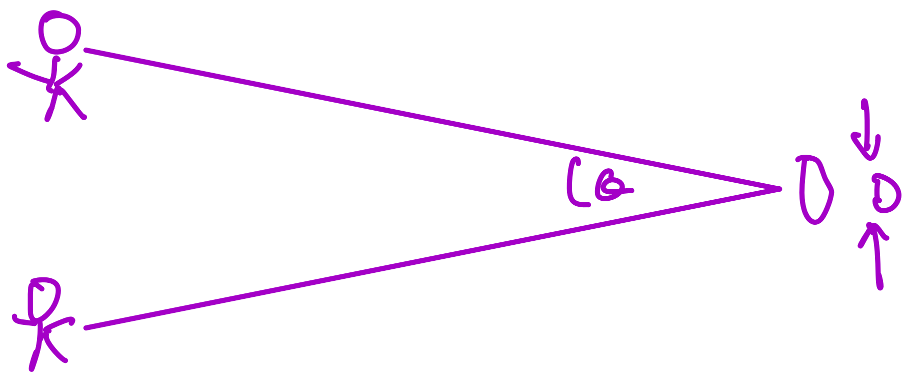
1st image falls on 2nd max of 2nd image - not resolvable

object angular separation is θ

if $\theta < \frac{1.22\lambda}{D}$ then the images are not resolvable

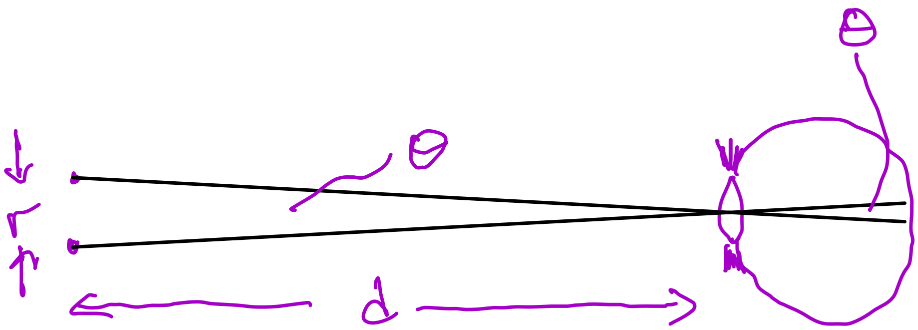
$\Rightarrow$ this is the diffraction limit for optical instruments

$\theta_{separation} > \frac{1.22\lambda}{D}$ to be resolvable



ex: eye pupil can be as small as 2mm

if light has $\lambda = 550$ nm, what is the minimum angle between 2 objects that you could see?



$$\theta = 1.22\frac{\lambda}{D} = \frac{1.22 * 550\times10^{-9}}{2\times10^{-3}} = 3.36\times10^{-4}$$

if head lights of a car are r=1.2m apart, what's the farthest dist the car can be for you to still resolve the 2 head lights (and not look like a single headlight)?

$$\theta = \frac{r}{d} = 3.36\times10^{-4}$$

$$d = \frac{r}{3.36\times10^{-4}} = \frac{1.2\text{m}}{3.36\times10^{-4}} = 3577\text{m}$$

~ 3.6km

~ 2.2miles

200 3/12 400 3/12

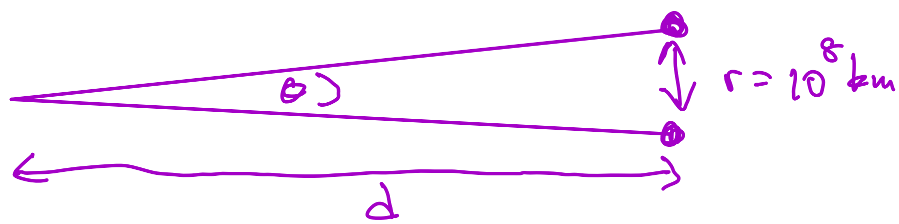
ex: eye pupil is ~0.4cm diameter dilated.

if 2 stars are 10^8 km apart (binary stars)

then what's the furthest distance they can be and still viewable by the eye without being "diffraction limited"?

use $\lambda = 400nm$

$$\theta = \frac{1.22\lambda}{D} = \frac{1.22 * 400\times10^{-9} m}{0.4\times10^{-2} m} = 1.22\times10^{-4}$$



$$\theta = \frac{r}{d} = 1.22\times10^{-4}$$

$$d = \frac{10^8 km}{1.22\times10^{-4}} = 8.2\times10^{11} km = 8.2\times10^{14} m$$

note: speed of light $c = 3\times10^8 m/s$

so the time to travel any dist $d = ct$

so the time for light to go $8.2\times10^{14} m$:

$$t = \frac{d}{c} = \frac{8.2\times10^{14} m}{3\times10^8 m/s} = 2732 \text{ light-sec}$$

$$= 45.5 \text{ light-min}$$

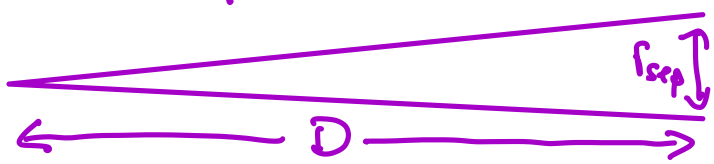
The closest star is Proxima Centauri

$d = 4.246$ light-years

$\Rightarrow$ whats the smallest separation at 4.246 light-yr that you could see a binary star or not? using 400 nm light?

diffraction condition: $\theta_{sep} \geq \theta_{diff}$

$\theta_{sep} = \frac{r_{sep}}{d}$



$\theta_{sep} = \frac{r_{sep}}{d} = \frac{1.22\lambda}{D}$ $\quad$ $D = 4$ mm eye pupil

$\sin\theta_{diff} \simeq \theta_{diff} = 1.22\lambda/D$ for small θ

so $\frac{1.22\lambda}{D} = \frac{r_{sep}}{d}$ solve for r_{sep}

$r_{sep} = \frac{1.22\lambda d}{D} = \frac{4.246 \text{ lt-yr} * 400\times10^{-9}\text{m}}{4\times10^{-3}\text{m}} = 4.246\times10^{-4}$ lt-yr

$\leftarrow$ pupil ~4mm

1 light-year = dist light goes in 1 yr = ct

so 1 lt-yr $= 3\times10^{8}\frac{m}{s} * 1\text{yr} * \frac{365\text{d}}{\text{yr}} * \frac{24\text{hr}}{\text{d}} * \frac{3600\text{sec}}{\text{hr}}$

$= 9.46\times10^{15}$ m

$r_{sep} = 4.246\times10^{-4}$ lt-yr $* \frac{9.46\times10^{15}\text{m}}{\text{lt-yr}} = 4.0\times10^{12}\text{m} = 4\times10^{9}$ km

Pluto orbit averages ~ 6×10^{9} km

Diffraction - "bending" waves

A shouts, sound hits opening.

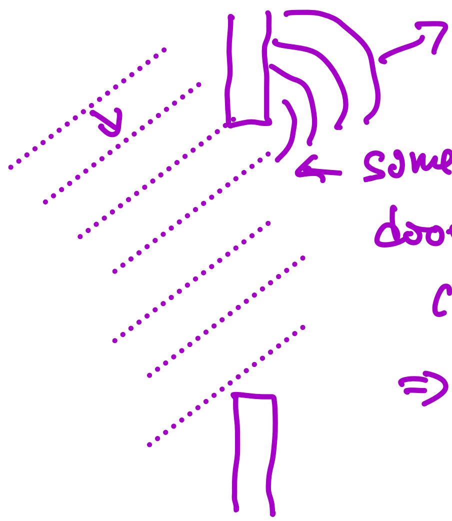
Note: $v_s = 343 \, m/s = \lambda f$

let $f = 200 \, Hz$

then $\lambda = \frac{v}{f} = \frac{343 \, m/s}{200/s} = 1.7 \, m = 5.6 \, ft$

door ways are ~4ft so $\lambda \sim a \Rightarrow$ diffraction

waves will hit "slit" and diffract around the ends



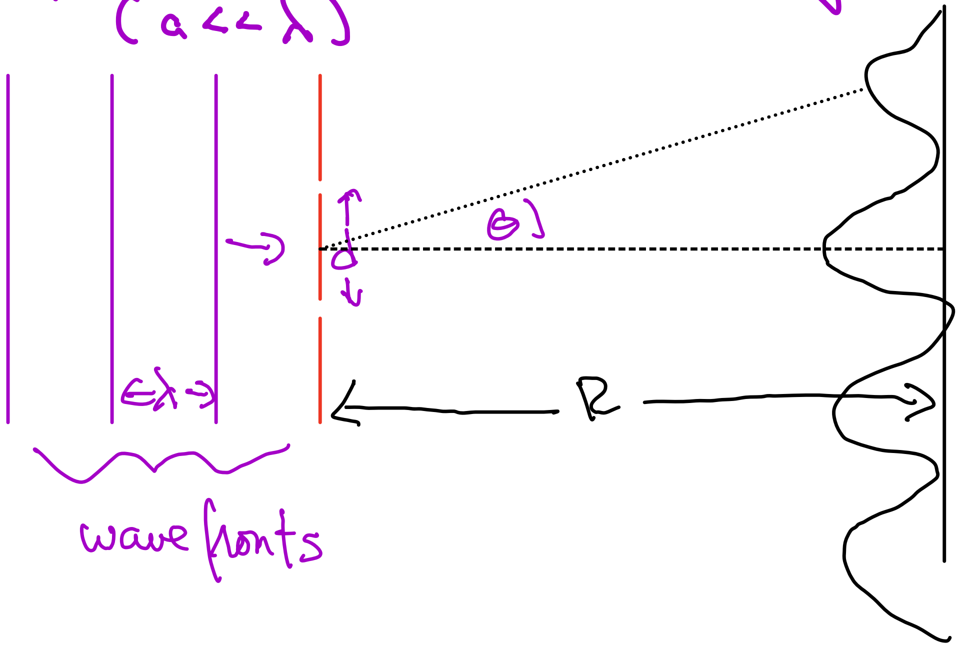
← some of Huygen's wavelets near door corner will be blocked causing diffraction

⇒ waves can "bend" around corners

Diffraction will always occur around edges!

2-slit Interference + Diffraction

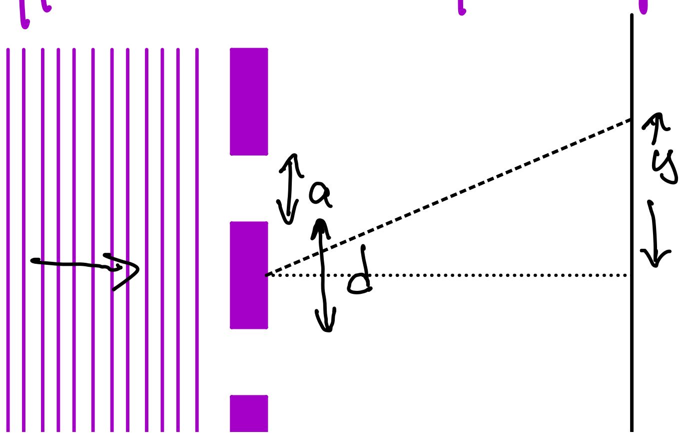
previous 2-slit interference ignored slit width ($a \ll \lambda$)



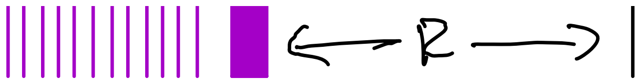
resulting interference pattern had maxima when $d\sin\theta = n\lambda$

$n = 0, \pm 1, \pm 2, \ldots$

If we use light, $\lambda \sim 550\,\text{nm}$, slits are usually with $a \gg \lambda$ so have to include diffraction in interference pattern



screen $R \gg a, d, \lambda$



2 slit interference: $d\sin\theta = n\lambda$ maxima
$= (n+\frac{1}{2})\lambda$ minima $\Big\} n = 0, \pm1, \pm2 ...$

diffraction: $a\sin\theta = m\lambda$ minima $m = \pm1, \pm2$

for $\sin\theta \sim$ small, $\tan\theta \approx \sin\theta \approx \theta$ and $\tan\theta = \frac{y}{R}$

2-slit Interference & Diffraction <u>both</u> happen
$\Rightarrow$ things are linear!

so position y for interference min:

$$d\frac{y_n}{R} = (n+\tfrac{1}{2})\lambda$$

or $$y_n = \frac{R\lambda}{d}(n+\tfrac{1}{2})$$

distance between interference minima:

$$\boxed{\Delta y_i = y_{n+1} - y_n = \frac{R\lambda}{d}}$$

position for diffraction minima:

$$\frac{a\,y_m}{R} = m\lambda$$

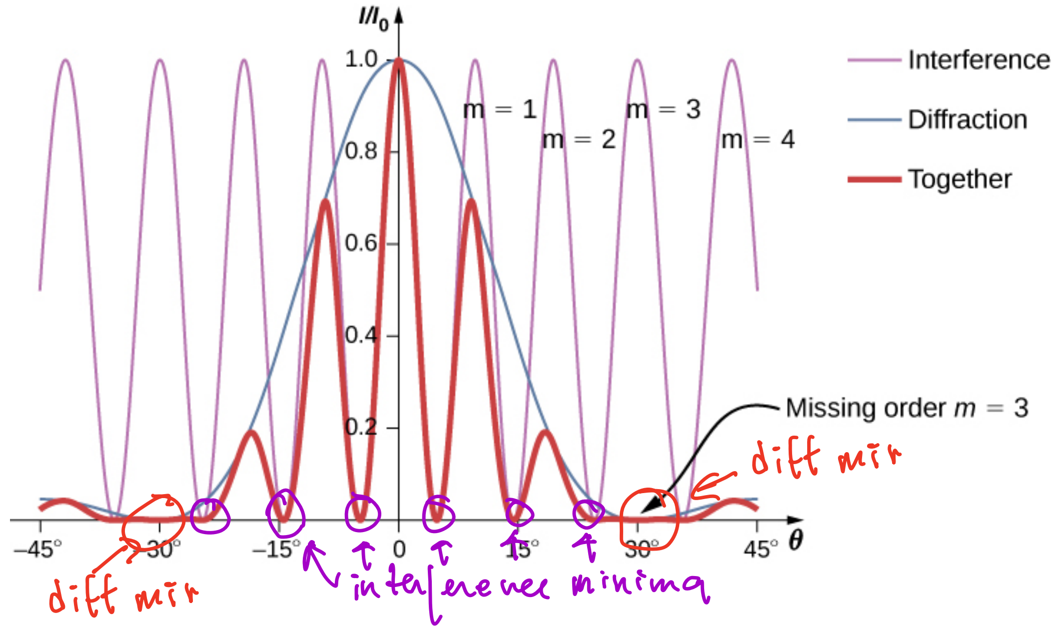
or $$y_m = \frac{R\lambda}{a}m$$

dist between diffraction minima:

$$\boxed{\Delta y_d = y_{m+1} - y_m = \frac{R\lambda}{a}}$$

usually $a < d$ so $\Delta y_d > \Delta y_i$

so there will be many interference minima inside diffraction minima



Intensity pattern for 2 slit:

$I = I_0 \cos^2 \phi/2$ where ϕ = phase diff between waves from top & bot slit

200 3/14

$\phi = k\Delta r = k * d\sin\theta$

$= \frac{2\pi d \sin\theta}{\lambda}$

Intensity for diffraction of single slit:

$$I = I_0 \, \text{sinc}^2\left(\frac{\pi a \sin\theta}{\lambda}\right)$$

overall intensity: diffraction modulates interference

$$I = I_0 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right) \text{sinc}^2\left(\frac{\pi a \sin\theta}{\lambda}\right)$$

interference maxima — diffraction minima

ex: $a = 1\text{ mm}$, $d = 3\text{ mm}$

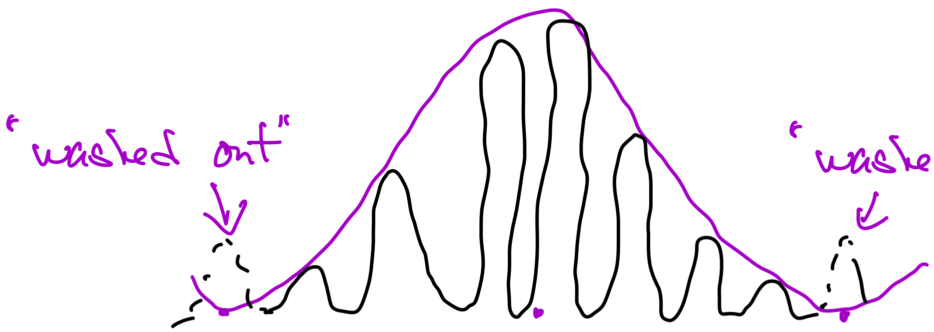
interference max is a pts $y_n = n\frac{\lambda R}{d}$

diffraction min " $y_m = m\frac{\lambda R}{a}$

since $d = 3a$ can write $y_n = n\frac{\lambda R}{3a}$

so int max coincides w/ diff min when $m = n/3$

this means 3rd interference maxima is washed out by 1st diffraction minima



⇒ in general, # interference maxima between diffraction minima will be

$$N = \underbrace{\frac{d}{a}}_{+\text{side}} + \underbrace{\frac{d}{a}}_{-\text{side}} - 1 \qquad = \frac{2d}{a} - 1$$

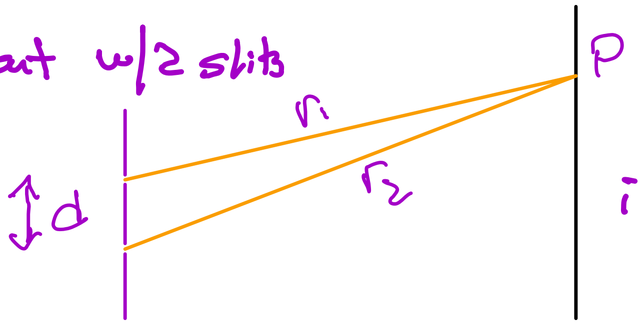
central max count once

$$= 2\frac{d}{a} - 1$$

$$= \frac{2d - a}{d}$$

Multiple slits

start w/2 slits



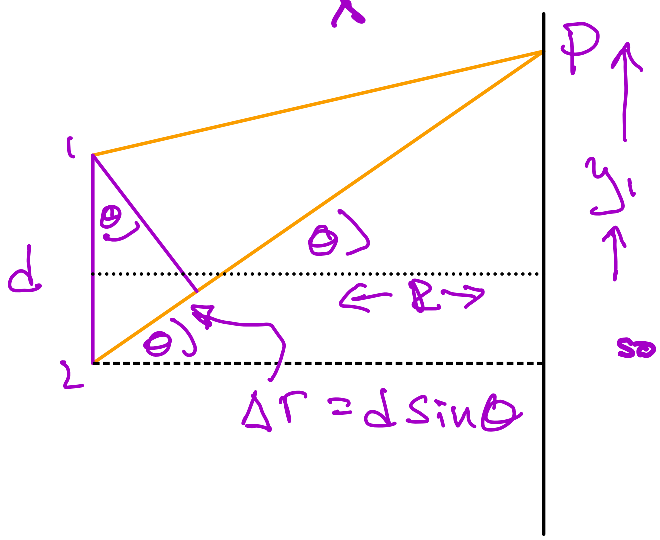
interference is constructive at P if $k\Delta r = 2\pi \cdot n$

$n = 0, \pm 1, \pm 2,$ etc.

for $n = 1$, then $k\Delta r = 2\pi$ $\qquad k = \frac{2\pi}{\lambda}$

and $\Delta r = d\sin\theta$

$$k\Delta r = \frac{2\pi d\sin\theta}{\lambda} = 2\pi \Rightarrow d\sin\theta = \lambda$$



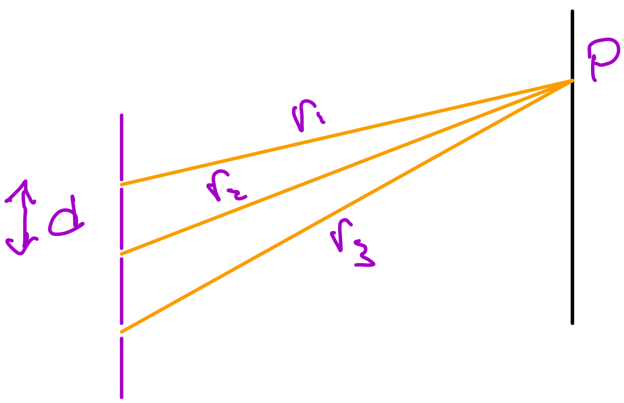
$\tan\theta = \frac{y_1}{R}$

and $\tan\theta \approx \sin\theta$

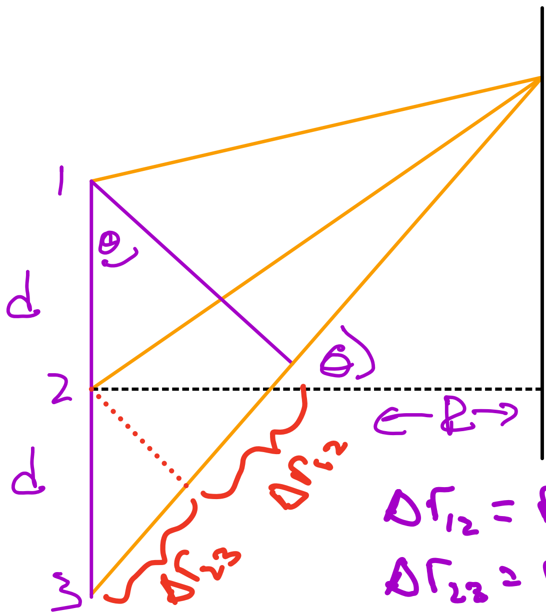
so $\sin\theta = \frac{\lambda}{d} = \frac{y_1}{R}$

$y_1 = \frac{\lambda R}{d}$

y_1 = height above line of symmetry between wave 1 & 2 now add another slit w/same spacing



this wave will also add constructively at P because the path diff to the other 2 waves will also be a multiple of λ



$\Delta r_{12} = r_2 - r_1 = d\sin\theta = \lambda$

$\Delta r_{23} = r_3 - r_2 = d\sin\theta = \lambda$

$\Delta r_{13} = r_3 - r_1 = (2d)\sin\theta = 2\lambda$

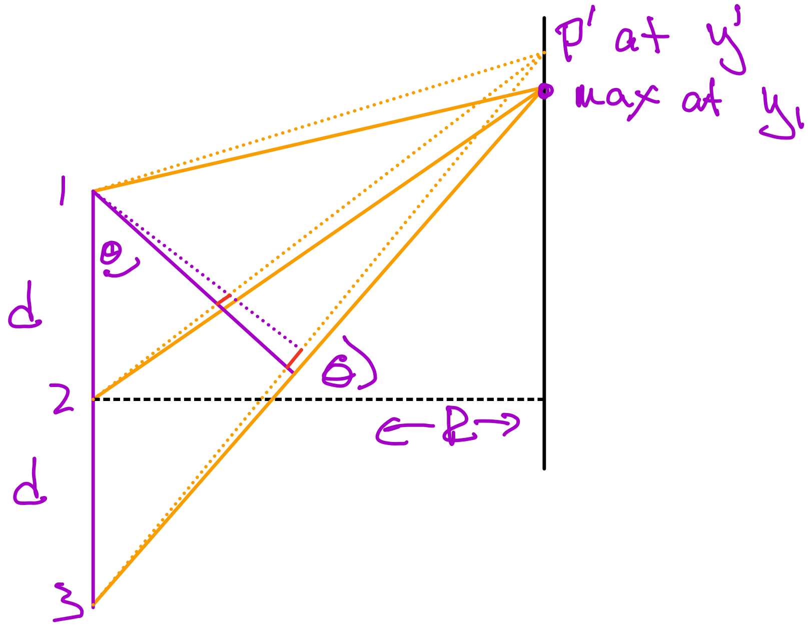
this is in the limit $R \gg\gg d$ (R=dist to screen)
so that all rays make an angle θ to the horizontal dashed line

$\Rightarrow$ each path diff has to be $\Delta r = \lambda$ for constructive interference for ALL waves with each other!

Next add more slits. Each slit will add constructively at point P with other sources (at the other slits)

This will produce a very bright max at point P

Now move point P slightly up from the max and add more slits:



Dashed lines are point P', at $y' > y_1$

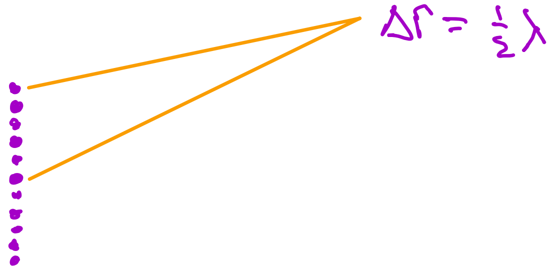
1st 2 waves have path diff Δr_{12} that is slightly bigger than before (red line)

2nd pair has a path difference Δr_{23} that is even <u>bigger</u> than Δr_1

$$\Delta r_{23} > \Delta r_1$$

each additional slit will have an even <u>bigger</u> path difference

for large enough number of slits, at some point the extra path difference will start to be $\frac{1}{2}\lambda$ from the 1st and will cancel each other out

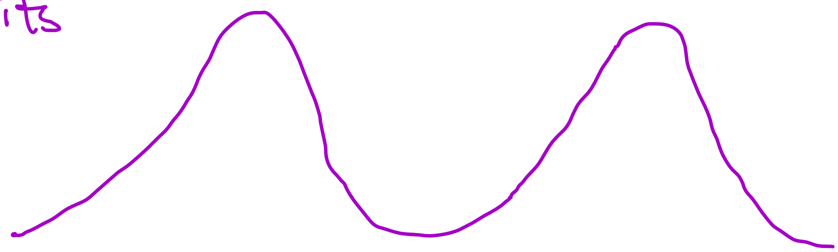


so right next to 1st max we will fall quickly to zero amplitude due to all the cancellations

$\Rightarrow$ for n w/ dist d between you will still see max when $d \sin\theta_n = n\lambda$

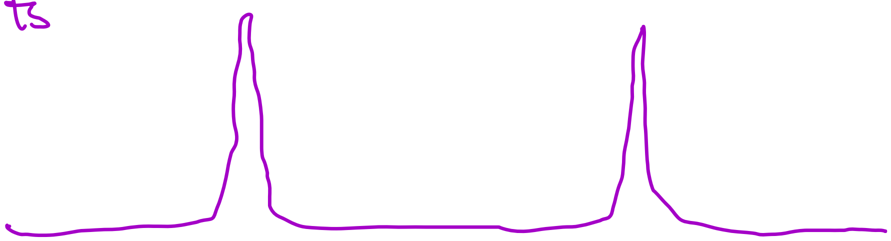
but as $m \to \infty$ the amplitude falls off more quickly

2 slits

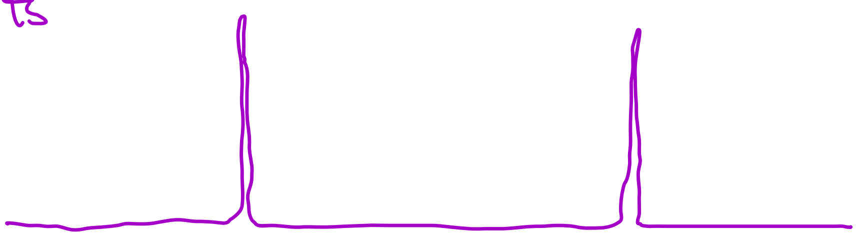


3 slits

10 slits

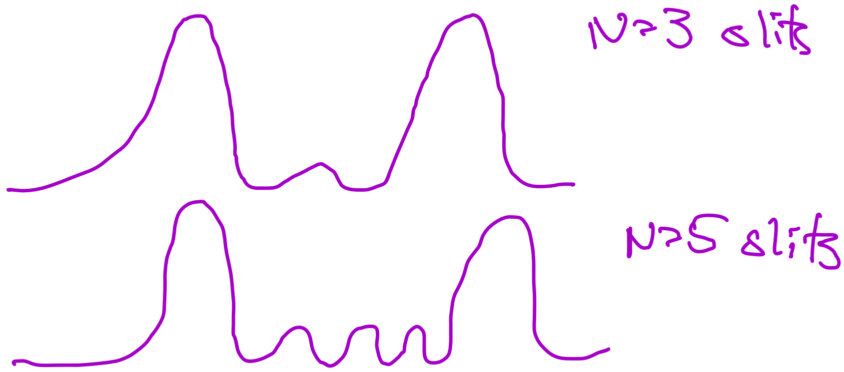


100 slits



Actually there is some structure between these maxima but it is much reduced
⇒ there are N-1 minima in between maxima

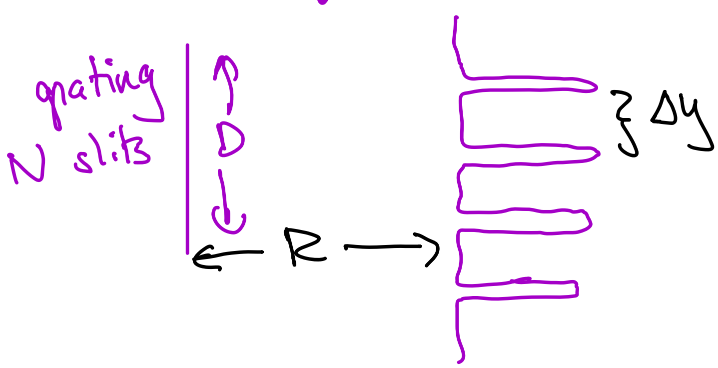
for N slits



but the slits all have to have the same separation!

Diffraction grating

This is used for very many slits (1000s)
The interference maxima are very peaked!



N slits per length D $\therefore$ spacing $d = \frac{D}{N}$

interference max are when $d \sin\theta = n\lambda$

and $\tan\theta \simeq \sin\theta = \frac{y_n}{R}$

so $d \frac{y_n}{R} = n\lambda$

so $y_n = n \frac{\lambda R}{d}$

then distance between maxima on the screen $\Delta y = y_{n+1} - y_n = \frac{\lambda R}{d}$

so if we measure $\Delta y, d, R$ carefully then $\lambda = \frac{d \cdot \Delta y}{R}$ tells you the wave length of light

$\Rightarrow$ Diffraction gratings can be used to measure wave lengths of light to high accuracy

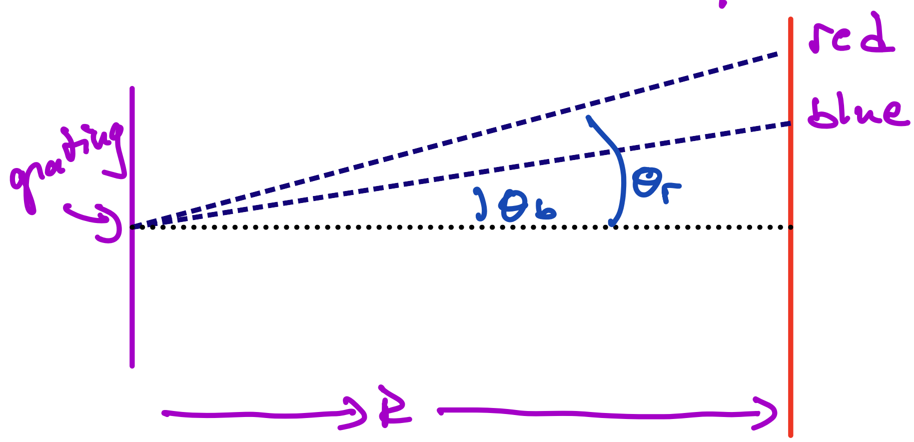
These can be made by etching lines on glass to be the slits.

ex: diffraction grating w/ 10,000 lines/cm

$\Rightarrow$ send beam of white light, screen $R = 2m$

Find angles for 380 nm (blue) and 760 nm (red)

note: all wavelengths have same condition for constructive interference



condition for constructive: $d \sin\theta = \lambda$

$\Rightarrow$ this is 1st max beyond central

note: all λ's interfere constructively at central max

so $d = \frac{1\,cm}{10{,}000} = 10^{-6} m$

$d \sin\theta_r = \lambda_r = 760\,nm$

$\sin\theta_r = \frac{760 \times 10^{-9}}{10^{-6}} = 0.76$

$\theta_r = \sin^{-1} 0.76 = 50^\circ$

$d \sin\theta_b = \lambda_b = 360$

so $\sin\theta_r = 0.38$

$\theta_r = 22^\circ$

200 3/24

400 3/24